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POST-BUCKLING BEHAVIOR OF CYLINDERS IN TORSION*

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ABSTRACT

The initial post-buckling behavior and the consequent imperfection-sensitivity of thin-walled cylinders subjected to torsion are studied on the basis of the Kármán-Donnell equations. A perturbation analysis consistent with the general theory of the post-buckling behavior of structures leads to eighth-order systems of complex, ordinary differential equations that are solved numerically for several sets of boundary conditions.

INTRODUCTION

Many studies concerned with the post-buckling behavior of shells have had as their aim the determination of the magnitudes of external loads associated with buckles of finite depth. The discovery, usually by approximate energy methods, of post-buckling loads smaller than the classical, initial buckling loads has justifiably been regarded as evidence that the classical loads may constitute unconservative estimates of buckling strength. Occasionally, calculations by similar approximate techniques have also been made for shells having assumed imperfections. The torsion of circular

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cylinders, perfect and imperfect, has been studied in this way by Loo [1] and Nash [2].

A different approach, pioneered by Koiter [3], [4], seeks to determine the initial post-buckling behavior; an asymptotically exact calculation then requires only the solution of linear problems to ascertain whether the applied load increases or decreases immediately after buckling. A drop in load after buckling of a perfect shell implies that the shell is imperfection-sensitive in the sense that a small initial geometrical imperfection in the shell would cause it to undergo snap-buckling at a load that is lower than the classical buckling load. Several shell problems have recently been treated in this way[5-9]; this series of studies is continued herein with a somewhat new twist, since torsion has not been considered previously. The shell theory used as a basis for the analysis is that attributed in [10] to Donnell, Mushtari, and Vlasov; as applied to a circular cylinder, the governing differential equations, when written in terms of a normal displacement and an Airy stress function, are usually called the Kármán-Donnell equations.

In order to make this paper self-contained, and to provide a convenient compendium of formulas for future use, the general theory of the post-buckling behavior of structures [1] will be quickly redeveloped for Donnell-Mushtari Vlasov shell theory. With a slight change of viewpoint (see [10]) the results will also be applicable to shallow-shell theory.

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POST-BUCKLING THEORY

The equilibrium relations of the non-linear shell theory of Donnell-Mushtari-Vlasov follow from the variational statement

$$\iint_A (M^{\alpha\beta}\delta K_{\alpha\beta} + N^{\alpha\beta}\delta E_{\alpha\beta})dA = E\ V\ W \tag{1}$$

where $M^{\alpha\beta}$ and $N^{\alpha\beta}$ are symmetrical stress-couples and stress-resultants, respectively; $K_{\alpha\beta}$ and $E_{\alpha\beta}$ are bending and membrane strains defined as

$$K_{\alpha\beta} = - W_{,\alpha\beta} \tag{2}$$

$$E_{\alpha\beta} = \frac{1}{2}[U_{\alpha,\beta} + U_{\beta,\alpha}] + b_{\alpha\beta}\ W + \frac{1}{2}\ W_{,\alpha}W_{,\beta} \tag{3}$$

in terms of the tangential displacements U_α and the normal displacement W ; and $b_{\alpha\beta}$ is the curvature tensor of the undeformed shell*. Commas denote covariant differentiation with respect to general surface coordinates ξ^1 , ξ^2 . The right-hand side of (1) represents the external virtual work of prescribed surface and edge loads, assumed constant-directional in this paper. The assertion (1) is supposed to hold for all δU_α and δW that do not violate boundary conditions; the calculus of variations then leads to the equilibrium equations

$$M^{\alpha\beta}{}_{,\alpha\beta} - b_{\alpha\beta}\ N^{\alpha\beta} + N^{\alpha\beta}W_{,\alpha\beta} - p = 0 \tag{4}$$

$$N^{\alpha\beta}{}_{,\alpha} = 0 \tag{5}$$

where p is external (inward) pressure, and tangential surface loads are assumed to be absent.

By writing the stress-resultants in terms of an Airy stress function F as

$$N^{\alpha\beta} = \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma}F_{,\omega\gamma} \tag{6}$$

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* Here $b_{\alpha\beta}$ is defined (unconventionally) as $\dfrac{\partial\vec{X}}{\partial\xi^\alpha}\ \dfrac{\partial\vec{N}}{\partial\xi^\alpha}$ in terms of the position vector $\vec{X}(\xi^1,\xi^2)$ and the unit (outward) surface normal $\vec{N}(\xi^1,\xi^2)$.

where $\varepsilon^{\alpha\omega}$ is the alternating tensor, Eqs.(5) are satisfied identically*. Then, with the use of the usual constitutive relations

$$\left.\begin{aligned} M^{\alpha\beta} &= \frac{Et^3}{12(r\nu^2)} \left[(1-\nu)K^{\alpha\beta} + \nu\, K^{\gamma}_{\gamma}\, g^{\alpha\beta}\right] \\ N^{\alpha\beta} &= \frac{Et}{1-\nu^2} \left[(1-\nu)E^{\alpha\beta} + \nu\, E^{\gamma}_{\gamma}\, g^{\alpha\beta}\right] \end{aligned}\right\} \tag{7}$$

where E is Young's modulus, ν is Poisson's ratio, t is the shell thickness, and $g^{\alpha\beta}$ is the metric tensor, Eqs.(4), (2), and (3) provide the equilibrium and compatibility equations

$$D\,\nabla^4 W + \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} b_{\alpha\beta} F_{,\omega\gamma} + p = \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} F_{,\omega\gamma} W_{,\alpha\beta} \tag{8}$$

$$\left(\frac{1}{Et}\right)\nabla^4 F - \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma}\, b_{\alpha\beta}\, W_{,\omega\gamma} = -\frac{1}{2}\,\varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} W_{,\alpha\beta}\, W_{,\omega\gamma} \tag{9}$$

where

$$D = \frac{Et^3}{12(1-\nu^2)} .$$

In the case of cylindrical shells, Eqs.(8) and (9) become the familiar Karman-Donnell equations.

It will be assumed that the prescribed loading is such that a membrane state governed by linear theory is always available as a solution (at least approximately). Thus, if λ denotes a scalar measure of the magnitude of the external loading, it is stipulated that

$$\left.\begin{aligned} N^{\alpha\beta} &= \lambda\, \overset{o}{N}{}^{\alpha\beta} = \lambda\, \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma}\, \overset{o}{F}_{,\omega\gamma} \\ M^{\alpha\beta} &= 0 \\ W_{,\alpha} &= 0 \end{aligned}\right\} \tag{10}$$

satisfies the variational equation (1). Hence, this variational equation

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* This is strictly correct only if the indices denoting successive covariant differentiation may be interchanged, but such interchange is legitimate in shells of zero Gaussian curvature, and in other shells introduces errors no larger than those already inherent in Donnell-Mushtari-Vlasov theory. (See [11] for discussion of this point.) This remark applies also to the derivation of Eq.(9).

can be written as

$$\iint_A [M^{\alpha\beta}\delta K_{\alpha\beta} + N^{\alpha\beta}\delta e_{\alpha\beta} + N^{\alpha\beta}W_{,\alpha}\delta W_{,\beta}]dA \qquad\qquad$$
$$- \lambda \iint_A \overset{o}{N}{}^{\alpha\beta}\delta e_{\alpha\beta}dA \tag{11}$$

where

$$e_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha}) + b_{\alpha\beta}W \tag{12}$$

is the linear part of $E_{\alpha\beta}$. Now suppose that a bifurcation off the equilibrium state (10) can occur at the critical load λ_c , and write the expansion

$$\begin{bmatrix} U_\alpha \\ W \\ N^{\alpha\beta} \\ F \\ M^{\alpha\beta} \\ K_{\alpha\beta} \\ E_{\alpha\beta} \\ e_{\alpha\beta} \end{bmatrix} = \lambda \begin{bmatrix} \overset{o}{U}_\alpha \\ \overset{o}{W} \\ \overset{o}{N}{}^{\alpha\beta} \\ \overset{o}{F} \\ 0 \\ 0 \\ \overset{o}{E}_{\alpha\beta} \\ \overset{o}{e}_{\alpha\beta} \end{bmatrix} + \varepsilon \begin{bmatrix} \overset{(1)}{U}_\alpha \\ \overset{(1)}{W} \\ \overset{(1)}{N}{}^{\alpha\beta} \\ \overset{(1)}{F} \\ \overset{(1)}{M}{}^{\alpha\beta} \\ \overset{(1)}{K}_{\alpha\beta} \\ \overset{(1)}{E}_{\alpha\beta} \\ \overset{(1)}{e}_{\alpha\beta} \end{bmatrix} + \varepsilon^2 \begin{bmatrix} \overset{(2)}{U}_\alpha \\ \overset{(2)}{W}_\alpha \\ \overset{(2)}{N}{}^{\alpha\beta} \\ \overset{(2)}{F} \\ \overset{(2)}{M}{}^{\alpha\beta} \\ \overset{(2)}{K}_{\alpha\beta} \\ \overset{(2)}{E}_{\alpha\beta} \\ \overset{(2)}{e}_{\alpha\beta} \end{bmatrix} + \ldots \tag{13}$$

where $\lambda \to \lambda_c$ as $\varepsilon \to 0$; the second column on the right is the classical buckling mode, assumed unique, and normalized in magnitude in some definite way; and the succeeding members of the expansion are all orthogonal to this buckling mode in some predetermined fashion.* Thus, in a well-defined manner, ε can be considered to represent the contribution of the classical buckling mode to the buckled state.

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* The only restriction that has to be placed on the choice of orthogonality condition is that it must not be possible for the buckling mode to be orthogonal to itself!

The elements <u>within</u> each of the columns of (13) are constrained by the <u>linear</u> relations (2), (6), (7) and (12). The non-linear strain displacement equation (3) couples the columns, since

$$
\begin{aligned}
\overset{o}{E}_{\alpha\beta} &= \overset{o}{e}_{\alpha\beta} \\
\overset{(1)}{E}_{\alpha\beta} &= \overset{(1)}{e}_{\alpha\beta} \\
\text{but} \qquad \overset{(2)}{E}_{\alpha\beta} &= \overset{(2)}{e}_{\alpha\beta} + \frac{1}{2}\overset{(1)}{W}_{,\alpha}\overset{(1)}{W}_{,\beta}
\end{aligned}
\tag{14}
$$

and so on. Finally, the variational statement (11), reduced by the use of (10), gives

$$
\begin{aligned}
&\iint_A [\overset{(1)}{M}{}^{\alpha\beta}\delta K_{\alpha\beta} + \overset{(1)}{N}{}^{\alpha\beta}\delta e_{\alpha\beta} + \lambda\, \overset{o}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha}\delta W_{,\beta}]dA \\
&+ \varepsilon\iint_A [\overset{(2)}{M}{}^{\alpha\beta}\delta K_{\alpha\beta} + \overset{(2)}{N}{}^{\alpha\beta}\delta e_{\alpha\beta} + (\lambda\overset{o}{N}{}^{\alpha\beta}\overset{(2)}{W}_{,\alpha} + \overset{(1)}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha})\delta W_{,\beta}]dA \\
&+ \varepsilon^2\iint_A [\overset{(3)}{M}{}^{\alpha\beta}\,\delta\, K_{\alpha\beta} + \overset{(3)}{N}{}^{\alpha\beta}\delta\; e_{\alpha\beta} + (\lambda\overset{o}{N}{}^{\alpha\beta}\overset{(3)}{W}_{,\alpha} + \overset{(1)}{N}{}^{\alpha\beta}\overset{(2)}{W}_{,\alpha} + \overset{(2)}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha})\delta W_{,\beta}]dA \\
&+ \ldots = 0
\end{aligned}
\tag{15}
$$

A variational equation for the buckling mode and load, obtained from (15) by letting $\varepsilon \to 0$, is

$$
\iint_A [\overset{(1)}{M}{}^{\alpha\beta}\delta K_{\alpha\beta} + \overset{(1)}{N}{}^{\alpha\beta}\delta e_{\alpha\beta} + \lambda_c\overset{o}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha}\delta W_{,\beta}]dA = 0
\tag{16}
$$

and a consequence of this relation is the "energy" equation

$$
\iint_A [\overset{(1)}{M}{}^{\alpha\beta}\overset{(1)}{K}_{\alpha\beta} + \overset{(1)}{N}{}^{\alpha\beta}\overset{(1)}{e}_{\alpha\beta}]dA = -\lambda_c\iint_A \overset{o}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha}\overset{(1)}{W}_{,\beta}dA \; .
\tag{17}
$$

The aforementioned orthogonality condition will be chosen as

$$
\iint_A [\overset{(1)}{M}{}^{\alpha\beta}\overset{(j)}{K}_{\alpha\beta} + \overset{(1)}{N}{}^{\alpha\beta}\overset{(j)}{e}_{\alpha\beta}]dA = -\lambda_c\iint_A \overset{o}{N}{}^{\alpha\beta}\overset{(1)}{W}_{,\alpha}\overset{(j)}{W}_{,\beta}dA = 0
\tag{18}
$$

for $j \geq 1$; by (16), it also holds for $j = 0$.

Note that the constitutive equations (7) imply the symmetry relations

$$\left.\begin{aligned} \overset{(i)}{M}{}^{\alpha\beta}\,\overset{(j)}{K}_{\alpha\beta} &= \overset{(j)}{M}{}^{\alpha\beta}\,\overset{(i)}{K}_{\alpha\beta} \\ \overset{(i)}{N}{}^{\alpha\beta}\,\overset{(j)}{E}_{\alpha\beta} &= \overset{(j)}{N}{}^{\alpha\beta}\,\overset{(i)}{E}_{\alpha\beta} \end{aligned}\right\} \tag{19}$$

for all i and j. With the use of (17), (18), and (19) the choice $\delta U_\alpha = \overset{(1)}{U}_\alpha$ and $\delta W = \overset{(1)}{W}$ in (15) then gives

$$(\lambda - \lambda_c) \iint_A \overset{o}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA + \frac{3\varepsilon}{2} \iint_A \overset{(1)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA$$

$$+ \; \varepsilon^2 \iint_A [2\overset{(1)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(2)}{W}_{,\beta} + \overset{(2)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,]dA + \ldots = 0 \tag{20}$$

Hence

$$\frac{\lambda}{\lambda_c} = 1 + a\,\varepsilon + b\,\varepsilon^2 + \ldots \tag{21}$$

where

$$a = \frac{3}{2\lambda_c} \; \frac{\iint_A \overset{(1)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA}{\iint_A \overset{o}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA} \tag{22}$$

$$b = -\frac{1}{\lambda_c} \; \frac{\iint_A [2\overset{(1)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(2)}{W}_{,\alpha} + \overset{(2)}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,]dA}{\iint_A \overset{o}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA} \tag{23}$$

A variational equation governing the elements of the ε^2 column in (13) can easily be written, but will not be needed. Instead, it is easier to work directly with the perturbation equations for $\overset{(i)}{W}$ and $\overset{(i)}{F}$ $(i = 1,2,\ldots)$ that can be derived directly from (8) and (9) as

$$\left.\begin{aligned} & D\,\nabla^4 \overset{(1)}{W} + \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} b_{\alpha\beta} \overset{(1)}{F}_{,\omega\gamma} - \lambda_c \overset{o}{N}{}^{\alpha\beta} \overset{(1)}{W}_{,\alpha\beta} = 0 \\ & \left(\frac{1}{Et}\right) \nabla^4 \overset{(1)}{F} - \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} \overset{(1)}{W}_{,\omega\gamma} = 0 \end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned} & D\,\nabla^4 \overset{(2)}{W} + \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} b_{\alpha\beta} \overset{(2)}{F}_{,\omega\gamma} - \lambda_c \overset{o}{N}{}^{\alpha\beta} \overset{(2)}{W}_{,\alpha\beta} \\ & \qquad = \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} \overset{(1)}{F}_{,\omega\gamma} \overset{(1)}{W}_{,\alpha\beta} + a\,\lambda_c \overset{o}{N}{}^{\alpha\beta} \overset{(1)}{W}_{,\alpha\beta} \\ & \left(\frac{1}{Et}\right) \nabla^4 \overset{(2)}{F} - \varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} \overset{(2)}{W}_{,\omega\gamma} = -\frac{1}{2}\,\varepsilon^{\alpha\omega}\varepsilon^{\beta\gamma} \overset{(1)}{W}_{,\alpha\beta} \overset{(1)}{W}_{,\omega\gamma} \end{aligned}\right\} \qquad (25)$$

and so on.

The coefficient a depends only on $\overset{(1)}{W}$, $\overset{(1)}{F}$, and vanishes whenever post-buckling behavior is independent of the sign of the buckling mode. When $a = 0$, the initial post-buckling behavior depends on b, the evaluation of which requires the determination of $\overset{(2)}{W}$ and $\overset{(2)}{F}$.

If $a = 0$ and $b < 0$, the shell is imperfection-sensitive. This is demonstrable from a repetition of the analysis, with (3) replaced by

$$E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2} W_{,\alpha} W_{,\beta} + \frac{1}{2} (W_{,\alpha} \tilde{W}_{,\beta} + \tilde{W}_{,\alpha} W_{,\beta}) \qquad (26)$$

when $\tilde{W}$ is an initial normal displacement. If $\tilde{W}$ is chosen as $\tilde{\varepsilon} \overset{(1)}{W}$, a singular perturbation expansion of the form

$$\begin{aligned} \begin{bmatrix} W \\ F \end{bmatrix} &= \lambda \begin{bmatrix} \overset{o}{W} \\ \overset{o}{F} \end{bmatrix} + \varepsilon \begin{bmatrix} \overset{(1)}{W} \\ \overset{(1)}{F} \end{bmatrix} + \varepsilon^2 \begin{bmatrix} \overset{(2)}{W} \\ \overset{(2)}{F} \end{bmatrix} + \dots \\ & + \tilde{\varepsilon}\,\varepsilon \begin{bmatrix} \overset{(1,1)}{W} \\ \overset{(1,1)}{F} \end{bmatrix} + \tilde{\varepsilon}\,\varepsilon^2 \begin{bmatrix} \overset{(2,1)}{W} \\ \overset{(2,1)}{F} \end{bmatrix} + \dots \\ & + \tilde{\varepsilon}^2 \varepsilon \begin{bmatrix} \overset{(1,2)}{W} \\ \overset{(1,2)}{F} \end{bmatrix} + \dots \\ & + \dots = 0 \end{aligned} \qquad (27)$$

becomes appropriate, where $\lim_{\varepsilon\to o}[\lim_{\tilde{\varepsilon}\to o}\lambda] = \lambda_c$ but $\lim_{\varepsilon\to o}\lambda = 0$ for $\tilde{\varepsilon} \neq 0$. It is then found that

$$(1 - \lambda/\lambda_c)\varepsilon + a\,\varepsilon^2 + b\,\varepsilon^3 + \ldots = (\lambda/\lambda_c)\tilde{\varepsilon} + \ldots . \tag{28}$$

Sketches of λ/λ_c vs. ε , given by (21) and (28) with $a = 0$ and $b < 0$, are shown in Figure (1) for $\tilde{\varepsilon} = 0$ and $\tilde{\varepsilon} \neq 0$. Let λ_s denote the maximum value of λ in the case $\tilde{\varepsilon} \neq 0$; then, as Koiter just showed in [1] ,

$$\left(1 - \frac{\lambda_s}{\lambda_c}\right)^{3/2} = \frac{3\sqrt{3}}{2}\,|\tilde{\varepsilon}|\,\sqrt{-b}\left(\frac{\lambda_s}{\lambda_c}\right) \tag{29}$$

so that $\left(1 - \frac{\lambda_s}{\lambda_c}\right) = 0(\tilde{\varepsilon})^{2/3}$, and small initial imperfections can induce large reductions in buckling strength.

It may be desirable to calculate the post-buckling variation of λ with the generalized displacement defined by

$$\Delta = \frac{1}{\lambda}\iint_A (M^{\alpha\beta}\,K_{\alpha\beta} + N^{\alpha\beta}\,E_{\alpha\beta})\,dA \quad . \tag{30}$$

(The significance of Δ is that $(\lambda\Delta)$ represents the decrease in potential energy of the applied loads.) Now let

$$\Delta_o = \iint_A \overset{o}{N}{}^{\alpha\beta}\,\overset{o}{e}_{\alpha\beta}\,dA \tag{31}$$

and note that letting $\delta e_{\alpha\beta} = \overset{o}{e}_{\alpha\beta}$, $\delta W \to 0$ in (15) shows that

$$\iint_A \overset{(j)}{N}{}^{\alpha\beta}\,\overset{o}{e}_{\alpha\beta}\,dA = 0 \tag{32}$$

for all $j \neq 0$. Substitution of (13) into (30), and the use of (14), (17), (18), (19) leads to

$$\Delta = \lambda\Delta_o - \frac{\varepsilon^2}{2}\iint_A \overset{o}{N}{}^{\alpha\beta}\,\overset{(1)}{W}_{,\alpha}\,\overset{(1)}{W}_{,\beta}\,dA + \ldots \tag{33}$$

If $a = 0$, $\varepsilon^2 \approx (\lambda - \lambda_c)/(\lambda_c b)$, and therefore the initial post-buckling stiffness $d\lambda/d\Delta$ at $\lambda = \lambda_c$ is

$$\left(\frac{d\lambda}{d\Delta}\right)_{\lambda-\lambda_c} = [\Delta_o - \frac{1}{2b\lambda_c} \iint_A \overset{o}{N}{}^{\alpha\beta} \overset{(1)}{W}_{,\alpha} \overset{(1)}{W}_{,\alpha} \, dA]^{-1} .$$

Since $d\lambda/d\Delta = \Delta_o^{-1}$ before buckling, the ratio K of the initial post-buckling stiffness to the prebuckling stiffness is

$$K = \left[1 - \frac{\iint_A \overset{o}{N}{}^{\alpha\beta} \overset{(1)}{W}_{,\alpha} \overset{(1)}{W}_{,\alpha} \, dA}{2b\lambda_c \iint_A \overset{o}{N}{}^{\alpha\beta} \overset{o}{e}_{\alpha\beta} \, dA} \right]^{-1} . \tag{34}$$

All of the results obtained are applicable in shallow-shell theory, wherein the middle surface of the shell lies a distance $z(\xi^1,\xi^2)$ above a plane, the curvature tensor $b_{\alpha\beta}$ is taken as $-z_{,\alpha\beta}$, and the metric tensor is taken as that of the (ξ^1,ξ^2) system in the reference plane rather than in the surface.

CYLINDER ANALYSIS

Differential Equations

For circular cylinders under torsion (Figure 2) the Kármán-Donnell equations are

$$\left.\begin{aligned} D\,\nabla^4 W + \left(\tfrac{1}{R}\right) F_{,xx} &= S(F,W) \\ \nabla^4 F - \frac{Et}{R}\, W_{,xx} &= -\frac{Et}{2}\, S(W,W) \end{aligned}\right\} \tag{35}$$

where $S(P,Q) = P_{,xx} Q_{,yy} + P_{,yy} Q_{,xx} - 2P_{,xy} Q_{,xy}$. The average shear stress $\tau \equiv \frac{1}{t}(N_{xy})av.$ will be chosen to play the role of the generalized load λ; hence, with

$$\overset{o}{F} = -\tau xy \tag{36}$$

the appropriate perturbation expansion is

$$\begin{bmatrix} W \\ \\ F \end{bmatrix} = \tau \begin{bmatrix} 0 \\ \\ \overset{o}{F} \end{bmatrix} + \varepsilon \begin{bmatrix} \overset{(1)}{W} \\ \\ \overset{(1)}{F} \end{bmatrix} + \varepsilon^2 \begin{bmatrix} \overset{(2)}{W} \\ \\ \overset{(2)}{F} \end{bmatrix} + \dots \quad . \tag{37}$$

The buckling equations (24) become

$$\left.\begin{aligned} D \, \nabla^4 \overset{(1)}{W} + \frac{1}{R} \overset{(1)}{F}_{,xx} - 2 \tau_c \overset{(1)}{W}_{,xy} &= 0 \\ \nabla^4 \overset{(1)}{F} - \frac{Et}{R} \overset{(1)}{W}_{,xx} &= 0 \end{aligned}\right\} \tag{38}$$

and, with the anticipation that $a = 0$, Eqs.(25) are

$$\begin{aligned} D \, \nabla^4 \overset{(2)}{W} + \frac{1}{R} \overset{(2)}{F}_{,xx} - 2\tau_c \overset{(2)}{W}_{,xy} &= S(\overset{(1)}{F},\overset{(1)}{W}) \\ \nabla^4 \overset{(2)}{F} - \frac{Et}{R} \overset{(2)}{W}_{,xx} &= -\frac{Et}{2} S(\overset{(1)}{W},\overset{(1)}{W}) \quad . \end{aligned} \tag{39}$$

Let $\xi = x/L$, $\eta = y/L$, and write

$$\left.\begin{aligned} \overset{(1)}{W} &= t \, \mathrm{Re} \, [w_1(\xi)e^{i\alpha\eta}] \\ \overset{(1)}{F} &= \left(\frac{Et^2L^2}{R}\right) \mathrm{Re} \, [f_1(\xi)e^{i\alpha\eta}] \end{aligned}\right\} \tag{40}$$

wherein α is a circumferential wave number making $\alpha R/L$ an integer. It can now be verified that the coefficient a , given by Eq.(21), does, in fact, vanish. Substitution of (40) into (38) gives

$$\left.\begin{aligned} w_1'''' - 2\alpha^2 w_1'' - 2i\alpha \Lambda_c w_1' + \alpha^4 w_1 + 12 Z^2 f_1'' &= 0 \\ f_1'''' - 2\alpha^2 f_1'' + \alpha^4 f_1 - w_1'' &= 0 \end{aligned}\right\} \tag{41}$$

where Z is Batdorf's parameter [12]

$$Z = \frac{L^2}{Rt} \sqrt{1 - \nu^2} \tag{42}$$

and

$$\Lambda_c = \frac{\tau_c L^2 t}{D} \tag{43}$$

The functional forms of $S(\overset{(1)}{F},\overset{(1)}{W})$ and $S(\overset{(1)}{W},\overset{(1)}{W})$ indicate that the solution of (39) can be written

$$\overset{(2)}{W} = \frac{t}{Z}\sqrt{1-\nu^2}\,\{w_{20}(\xi) + \mathrm{Re}[w_{22}(\xi)e^{2i\alpha\eta}]\} \tag{44}$$

$$\overset{(2)}{F} = \frac{Et^2L^2}{RZ}\sqrt{1-\nu^2}\{f_{20}(\xi) + \mathrm{Re}[f_{22}(\xi)e^{2i\alpha\eta}]\}$$

and then, with a bar to denote complex conjugate,

$$w_{20}'''' + 12Z^2 f_{20}'' = -\,6\alpha^2Z^2\ \mathrm{Re}[(f_1w_1)''] \tag{45}$$

$$f_{20}'''' - w_{20}'' = \frac{\alpha^2}{4}\,(w_1\bar{w}_1)'' \tag{46}$$

$$\left.\begin{aligned} & w_{22}'''' - 8\alpha^2w_{22}'' - 4i\Lambda_c\alpha w_{22}' + 16\alpha^4 w_2 + 12Z^2 f_{22}'' \\ & \qquad = -\,6\alpha^2Z^2(f_1w_1'' + f_1''w_1 - 2f_1'w_1') \\ & f_{22}'''' - 8\alpha^2 f_{22}'' + 16\alpha^4 f_2 - w_2'' = \frac{\alpha^2}{2}[w_1w_1'' - (w_1')^2] \end{aligned}\right\} . \tag{47}$$

Note that a contribution to $\{\overset{(2)}{W},\overset{(2)}{F}\}$ proportional to $\{\overset{(1)}{W},\overset{(1)}{F}\}$ is ruled out by the orthogonality condition

$$\int_0^L dx \int_0^{2\pi R} dy[\overset{(1)}{W}_{,x}\overset{(2)}{W}_{,y} + \overset{(1)}{W}_{,y}\overset{(2)}{W}_{,x}] = 0 .$$

From the condition that the circumferential displacement V be single valued, it follows that

$$\int_0^{2\pi R} V_{,y}dy = \int_0^{2\pi R}[\frac{1}{Et}(N_y - \nu N_x) - \frac{W}{R} - \frac{1}{2}(W_{,y})^2]dy = 0$$

whence, in the absence of a net end thrust,

$$f_{20}'' = w_{20} + \frac{\alpha^2}{4}\,(w_1\bar{w}_1) \quad . \tag{48}$$

This is consistent with (46), and now f_{20}'' can be eliminated from (45), which becomes

$$w_{20}'''' + 12Z^2 w_{20} = -3\alpha^2 [w_1 \bar{w}_1 + 2\,\mathrm{Re}(f_1 w_1)''] \, . \qquad (49)$$

Note that w_{20} , f_{20} are real, whereas W_1 , f_1 , w_{22} , f_{22} are complex.

Boundary Conditions

Calculations will be made for the following three sets of boundary conditions at $x = 0,L$ (see Figure 2):

I. $W = W_{,xx} = 0$ (simple support)

$N_x = V_{,y} = 0 \; ; \quad \frac{1}{2\pi R} \oint_0^{2\pi R} N_{xy} dy = t\tau$

II. $W = W_{,x} = 0$ (clamped)

$N_x = V_{,y} = 0 \; ; \quad \frac{1}{2\pi R} \oint_0^{2\pi R} N_{xy} dy = t\tau$

III. $W = W_{,x} = 0$ (clamped)

$U_{,y} = V_{,y} = 0 \; ; \quad \frac{1}{2\pi R} \oint_0^{2\pi R} N_{xy} dy = t\tau \; ; \quad \oint_0^{2\pi R} N_x dy = 0 \, .$

These sets of conditions are designated S3 , C3 , and C1 by Yamaki and Kodama [13] in their recent study of torsional buckling.

With these boundary conditions, it is easily shown that there is no loss in generality in assuming that the real parts of all functions of ξ are symmetrical, and their imaginary parts antisymmetrical, in the interval $0 \leq \xi \leq 1$.

Buckling Problem

Donnell's early approximate solutions for torsional buckling of cylinders [14] were somewhat improved upon by Batdrof, Stein, and Schildcrout [12];

but the most accurate available solutions to the eigenvalue problem (41) for a variety of boundary conditions are the essentially exact ones presented by Yamaki and Kodama. In the present work, however, a unified numerical approach was used in both the buckling and post-buckling problems, and the buckling results of Yamaki and Kodama served as a check on the accuracy of the procedure.

A standard finite-difference scheme was used to approximate Eqs.(41). The consequent difference equations, together with appropriate versions of the boundary conditions I, II, and III in terms of w_1 and f_1, provided eigenvalue problems in matrix form that were easily solved with the help of complex arithmetic programmed by means of Fortran IV on an IBM 7094 computer. (Details of the finite-difference analysis are given in the Appendix.) The eigenvalues Λ_c were minimized with respect to the circumferential wave number α on the basis of the simplifying assumption that α could vary continuously; the results thus found for Λ_c and the minimizing values of α are tabulated in Tables I, II, and III for the three sets of boundary conditions considered. Comparison of these results with those of Yamaki and Kodama (who actually tabulate $k_s = \Lambda_s/\pi^2$ and $\beta = \alpha/\pi$) revealed discrepancies of only small fractions of one per cent*.

Plots of Λ_c vs. Z are shown in Figure 3. Yamaki and Kodama discuss the limitations on these results associated with the occurrence of small numbers of circumferential waves, and they also make inferences concerning the relative effects of various types of boundary constraint.

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* In case III, it is necessary to assume a value for Poisson's ratio. In this paper, $\nu = \frac{1}{3}$, whereas Yamaki and Kodama used $\nu = .3$; but they also made a study of the influence of ν which indicates that this small difference is quite negligible.

Post-buckling Problem $(\overset{(2)}{W}, \overset{(2)}{F})$

The differential equations (47) and (49) for w_{22} , f_{22} , and w_{20} were also discretized and, with Λ_c , α , and the non-homogeneous right-hand sides available from the solution of the buckling problem, were solved numerically. (Again, details are given in the Appendix.) In all cases, the eigenfunctions w_1 , f_1 used in the non-homogeneous terms were normalized to make $|w_1|_{max} = 1$.

Evaluation of Post-buckling Coefficient b

Invoking the initial stipulation that $\alpha R/L$ be an integer, and exploiting the symmetry properties of the buckling and post-buckling modes, permits the formula (23) for b to be transformed to

$$\frac{b}{1-\nu^2} = \frac{3\alpha}{\Lambda_c}\left\{\int_0^{1/2} \mathrm{Re}[-4\bar{f}_1 w_1' w'_{20} - 2f_1 w_1' \bar{w}_{22}' + 4f_1'' w_1 \bar{w}_{22} - 4f_1' w_1' \bar{w}_{22} - 4\bar{f}_1' w_1 w_{20}' + 2f_1' w_1 \bar{w}_{22}' - 4f_{22}(\bar{w}_1')^2 + 2f_{20}'' w_1 \bar{w}_1 - f_{22}''(\bar{w}_1)^2 - 4f_{22}' \bar{w}_1' \bar{w}_1]\, d\xi\right\}\left\{\int_0^{1/2} \mathrm{Im}(w_1 \bar{w}_1')\, d\xi\right\}^{-1} \tag{50}$$

in terms of the buckling solutions w_1 , f_1 and the post-buckling solutions w_{20} , w_{22} , and f_{22} . The function f_{20}'' in (50) is given by (48) in terms of w_{20} and w_1 . There is, of course, an approximation inherent in using these solutions, as well as the corresponding values of Λ_c and α , in the formula (50), since the requirement that $\alpha R/L$ be an integer was relaxed in their calculation.

The integrals in (50) were evaluated numerically; the results for $\frac{b}{1-\nu^2}$ are given in Tables I, II, III and are plotted against Z in Figure 4.

Evaluation of Post-buckling Coefficient K

Let γ be the apparent shear strain defined by $\gamma L = V(L) - V(0)$. Then, in the present problem, the post-buckling stiffness ratio K has the interpretation

$$K = \frac{1}{G}\left(\frac{d\tau}{d\gamma}\right) = \frac{d(\tau/\tau_c)}{d(\gamma/\gamma_c)} \tag{51}$$

where $G = \frac{E}{2(1+\nu)}$ is the shear modulus, and $\tau_c = G\gamma_c$. Eq.(39) can be manipulated into the form

$$K = \left[1 + \frac{6\alpha(1-\nu)}{\Lambda_c b} \int_0^{1/2} \mathrm{Im}\,(w_1 \bar{w}_1')\,d\xi\right]^{-1} \tag{52}$$

and, for the choice $\nu = \frac{1}{3}$, provides the results in tables I, II, III. Figure (5) shows a series of plots of τ/τ_c vs. γ/γ_c in which the initial post-buckling slopes K are shown to scale.

DISCUSSION OF RESULTS

It is found by interpolation that the post-buckling coefficient b is negative, and hence that imperfection-sensitivity exists, in the following ranges of Z :

$$\begin{aligned} &\text{Case I:} && Z > 2.0 \\ &\text{Case II:} && Z > 8.0 \\ &\text{Case III:} && Z > 9.7 \quad . \end{aligned} \tag{53}$$

In all three cases, the magnitude of b tends to zero as Z becomes very large. The magnitudes of the actual buckling stresses τ_s of the cylinders in these ranges of Z cannot, of course, be predicted, since they depend on the imperfections as well as on b . However, a rough indication of the extent to which buckling strengths might be degraded is afforded by Eq.(29). Because of the way the buckling mode $W^{(1)}$ was normalized, the imperfection parameter $\tilde{\varepsilon}$ in this equation can be identified with $\tilde{\delta}/t$, where $\tilde{\delta}$ is the maximum amplitude of that part of the initial deflection that is in the shape of this mode. The curves in Figure (6), based on Eq.(29), show how $\tau_s/\tau_c (= \lambda_s/\lambda_c)$ varies with $\tilde{\delta}/\tau$ for $b = -.01, -.1$, and -1. It may be recalled (see Figure 4) that for $Z > 100$, $(-b)$ remains less than .1, and only in case I does it ever exceed .2, near $Z = 10$. Accordingly, it might be deduced from Figure (6) that, with reasonable fabrication care to keep $\tilde{\delta}/t$ less than, say, .5, τ_s should rarely be less than about 70% of τ_c.

This is more or less consistent with the experimental data collected from various sources by Batdorf, Stein, and Schildcrout in [12] and reproduced in Figure (7) on a plot of $k_s = \Lambda_c/\pi^2$ vs Z for case I, together with the curve for b. Unfortunately, there is little data in the range of maximum imperfection-sensitivity predicted for $Z \sim 10$. (Admittedly, thin cylinder proportions get awkward at such values of Z.) Furthermore, one might expect to see a general improvement between classical theory and experiment as Z increases to large values, but this does not seem to occur. However, this kind of expectation may not be justified, since, as is evident from Eq.(29) and Figure (6), the effect of a beneficial decrease in the magnitude of $(-b)$ could easily be nullified by a simultaneous small increase in $\tilde{\delta}/t$.

The graphs in Figure (5) have an interesting consequence. In Case I there is evidently a small range of Z around $Z = 10$, in which the apparent shear strain γ, as well as the average shear stress τ, decreases after buckling. This does not occur in cases II and III. Whenever $b < 0$, snap buckling can be expected under a monotonically increasing torque; but if $bK<0$ snapping would also occur under an imposed monotonically increasing relative rotation of the cylinder ends. It might be amusing to try to verify this last prediction experimentally; but it should be noted that it applies only to perfect cylinders, and sufficiently large imperfections might prevent the occurrence of this kind of snapping.

ACKNOWLEDGEMENT

The writer is grateful to Mr. James C. Frauenthal for checking the cylinder analysis, writing the Fortan IV program, and calculating the numerical results.

APPENDIX

NUMERICAL ANALYSIS

Buckling Problem

The differential equations (41) were approximated, in the interval $0 \leq \xi \leq \frac{1}{2}$ by the N finite-difference matrix equations

$$\bar{A}(\alpha) z_{n-1} + B(\alpha) z_n + A(\alpha) z_{n+1} = 0 \qquad (n = 1,2,\ldots N) \tag{A1}$$

where

$$z = \begin{bmatrix} w_1 \\ w_1'' \\ f_1 \\ f_1'' \end{bmatrix} \tag{A2}$$

$$A(\alpha) = \begin{bmatrix} \Delta^{-2} & 0 & 0 & 0 \\ -i\alpha\Lambda_c\Delta^{-1} & \Delta^{-2} & 0 & 0 \\ 0 & 0 & \Delta^{-2} & 0 \\ 0 & 0 & 0 & \Delta^{-2} \end{bmatrix} \tag{A3}$$

$$B(\alpha) = \begin{bmatrix} -2\Delta^{-2} & -1 & 0 & 0 \\ \alpha^4 & -2(\alpha^2-\Delta^{-2}) & 0 & 12Z^2 \\ 0 & 0 & -2\Delta^{-2} & -1 \\ 0 & -1 & \alpha^4 & -2(\alpha^2-\Delta^{-2}) \end{bmatrix} \tag{A4}$$

$$\Delta = \frac{1}{2N} \tag{A5}$$

and the bar denotes complex conjugate. In Potters' method [15] (essentially Gaussian elimination) the equations

$$z_n = -P_n z_{n+1} \qquad (n = 0,1,2,\ldots N) \tag{A6}$$

are written, and then (A1) provides the recurrence matrix relation

$$P_n = [B - \bar{A} P_{n-1}]^{-1} A \qquad (n = 1,2,\ldots N) \quad . \tag{A7}$$

In Case I, the boundary conditions imply $w_1 = w_1'' = f_1 = f_1'' = 0$ at $\xi = 0$, whence $z_o = 0$. With the initial condition $P_o = 0$, (A7) is used to get $P_1, P_2, \ldots P_N$.

Without loss of generality, the symmetry conditions

$$\mathrm{Im}(z) = \mathrm{Re}(z') = 0 \qquad @\ \xi = \frac{1}{2}$$

are now imposed. These conditions imply

$$\left.\begin{array}{r} z_n - \bar{z}_n = 0 \\ z_{N+1} + \bar{z}_{N+1} - z_{N-1} - \bar{z}_{N-1} = 0 \end{array}\right\} \tag{A8}$$

from which it is deduced that

$$[P_N^{-1} + \bar{P}_N^{-1} - P_{N-1} - \bar{P}_{N-1}]\, z_N = 0 \quad . \tag{A9}$$

For a given α, the eigenvalue Λ_c must therefore satisfy the 4×4 determinatal equation

$$\left| P_N^{-1} + \bar{P}_{N-1} - P_{N-1} - \bar{P}_{N-1} \right| = 0 \quad . \tag{A10}$$

Once the eigenvalue Λ_c (minimized with respect to α) was found, z_N was calculated, to within an arbitrary factor, from (A9), and a "backward" sweep based on (A6) provided all the other z's making up the eigenmode. Then z was normalized to make $|w_1|_{max} = 1$.

For the other two cases, the boundary conditions on w_1, f_1 are easily shown to be:

Case II: $w_1 = w_1' = f_1 = f_1'' = 0$

Case III: $w_1 = w_1' = f_1'' + \nu\, \alpha^2 f_1 = f_1''' - (2 + \nu)\alpha^2 f_1' = 0$.

The only new things needed to handle these cases are appropriate initial values of the matrix P_o. In both cases the procedure used was to introduce a fictitious station at $n = -1$, write the boundary conditions in the form

$$- G\, z_{-1} + H\, z_o + G\, z_1 = 0 \tag{A11}$$

and also write the difference equation (A1) at $n = 0$. Then by elimination of z_{-1} it follows that

$$P_o = (H + G\, A^{-1}\, B)^{-1}\, (G + G\, A^{-1}\, A)\ . \tag{A12}$$

In Case II,

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A13}$$

and in Case III,

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -(2+\nu)\alpha^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \nu\alpha^2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A14}$$

<u>Post-buckling Problem</u>

The differential equations (47) are approximated by

$$A(2\alpha)\zeta_{n-1} + B(2\alpha)\zeta_n + A(2\alpha)\zeta_{n+1} = g_n \tag{A15}$$

with

$$\zeta = \begin{bmatrix} w_2 \\ w_2'' \\ f_2 \\ f_2'' \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ -\,6\alpha^2 Z^2 (f_1\, w_1'' + f_1''\, w_1 - 2f_1'\, w_1') \\ 0 \\ \frac{\alpha^2}{2}\, [w_1 w_1'' - (w_1')^2] \end{bmatrix} \tag{A16}$$

Potters' method now says

$$\zeta_n = -\, Q_n\, \zeta_{n+1} + \chi_n \tag{A17}$$

and (A15) gives

$$Q_n = [B(2\alpha) - \bar{A}(2\alpha)Q_{n-1}]^{-1} A(2\alpha) \tag{A18}$$

$$\chi_n = [B(2\alpha) - \bar{A}(2\alpha)Q_{n-1}]^{-1} [g_n - \bar{A}(2\alpha)\chi_{n-1}] \quad . \tag{A19}$$

The initial values Q_o are the same as those for P_o in the buckling problem, except that α is replaced by 2α wherever it appears; $\chi_o = 0$ in all cases. The symmetry conditions (A8) now give

$$\zeta_N = \left[Q_{N-1} + \bar{Q}_{N-1} - Q_N^{-1} - \bar{Q}_N^{-1}\right]^{-1} \left[\chi_{N-1} + \bar{\chi}_{N-1} - P_N^{-1}\chi_N - \bar{P}_N^{-1}\bar{\chi}_N\right] \tag{A20}$$

and then all the ζ's are found by means of a backward sweep based on (A17).

An obvious 2×2 matrix analogue of these procedures was used to calculate w_{20} from Eq.(49). The number of intervals used in the calculations varied from $N = 50$ to $N = 90$.

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TABLE I

Case I: $W = W_{,xx} = N_x = V_{,y} = 0$

Z	Λ_c	α	$\frac{b}{1-\nu^2}$	K
0	52.67	2.51	.2139	.671
1	53.18	2.57	.1508	.582
3	56.51	2.86	-.0832	-2.00
10	74.29	3.97	-.2364	3.42
30	124.5	5.90	-.1583	-7.51
100	271.6	9.29	-.0756	- .913
300	602.9	13.39	-.0342	- .407
1,000	1482.	19.75	-.0122	- .175
10,000	8389.	37.4	-.0015	- .031

TABLE II

Case II: $W = W_{,x} = N_x = V_{,y} = 0$

Z	Λ_c	α	$\frac{b}{1-\nu^2}$	K
0	88.58	3.79	.2513	.673
1	88.76	3.80	.2405	.663
3	89.88	3.87	.1694	.576
10	99.58	4.45	-.0682	- .938
30	141.9	6.13	-.1385	-6.23
100	286.4	9.44	-.0833	-1.212
300	617.6	13.78	-.0346	- .394
1,000	1502.	19.94	-.0124	- .179
10,000	8422.	37.5	-.0015	- .052

TABLE III

Case III: $W = W_{,x} = U_{,y} = V_{,y} = 0$

$$\left(\oint N_x dy = 0\right)$$

Z	Λ_c	α	$\frac{b}{1-\nu^2}$	K
0	88.58	3.79	.256	.677
1	88.78	3.79	.247	.669
3	90.01	3.88	.192	.606
10	100.8	4.49	-.0042	- .030
30	146.6	6.35	-.0961	-1.249
100	298.5	9.86	-.0742	- .797
300	648.4	14.69	-.0350	- .342
1,000	1593.	21.45	-.0135	- .176
10,000	9094.	41.9	-.0017	- .052

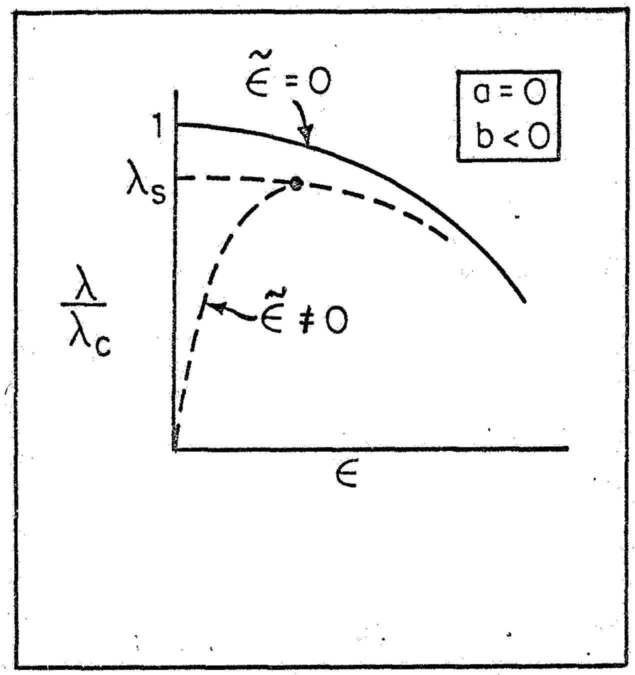

FIG. 1 INFLUENCE OF SMALL INITIAL IMPERFECTION

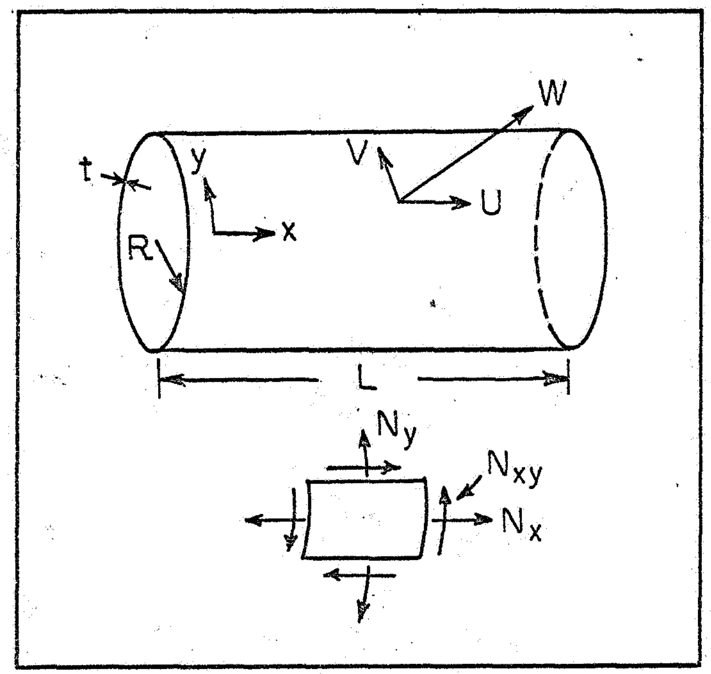

FIG. 2 CYLINDER NOTATION

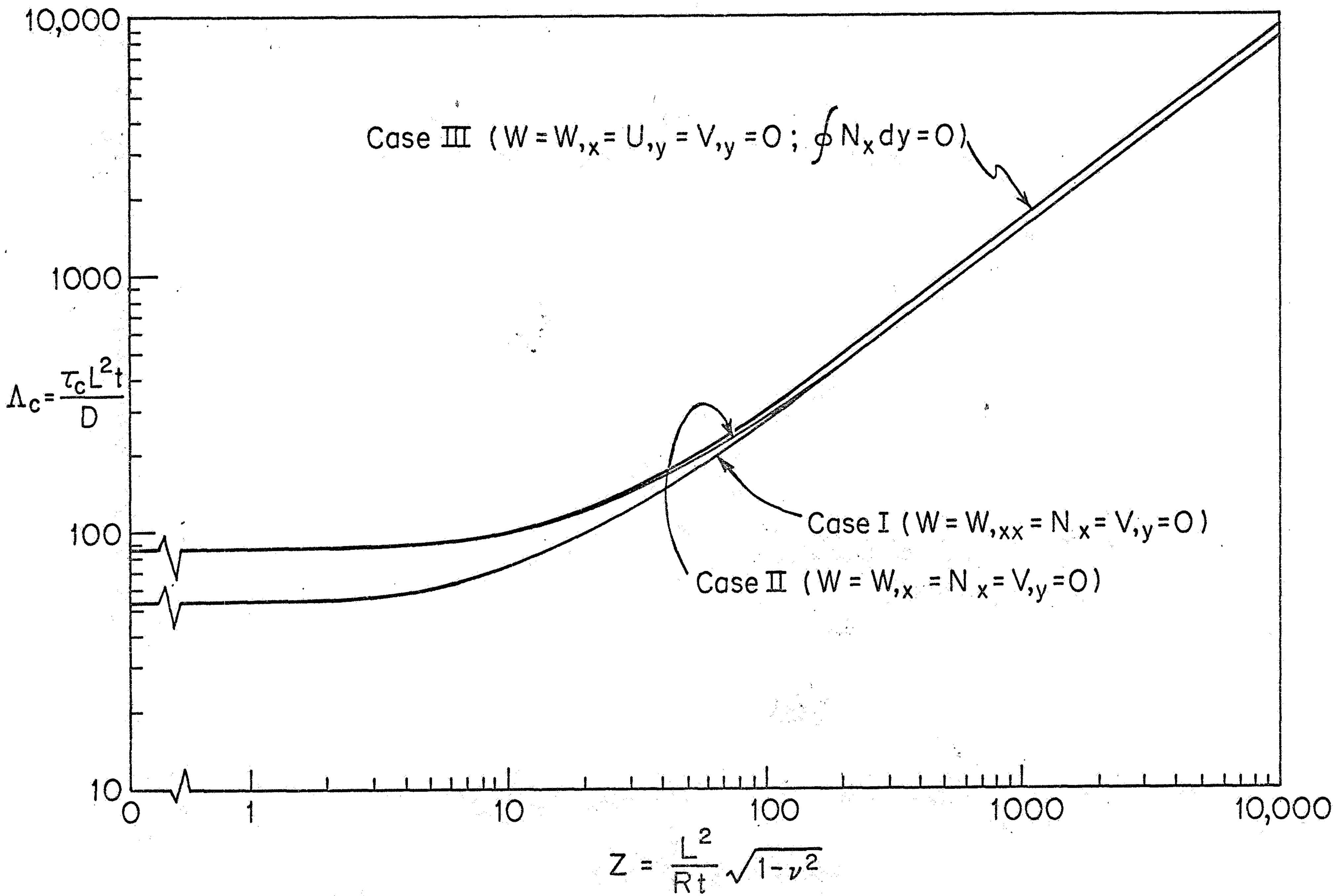

FIG. 3 TORSIONAL BUCKLING COEFFICIENT Λ_c

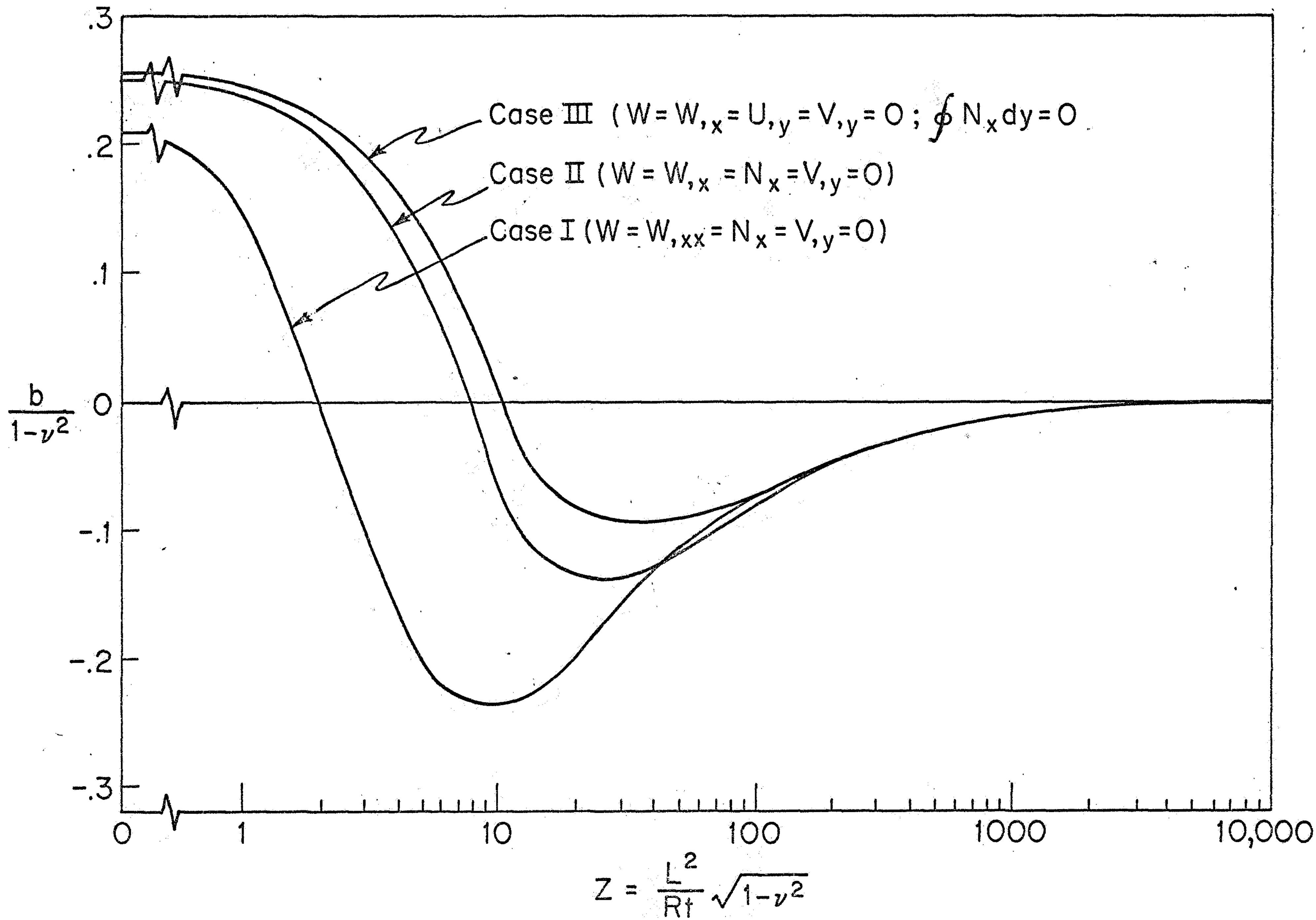

FIG. 4 POST-BUCKLING. COEFFICIENT b.

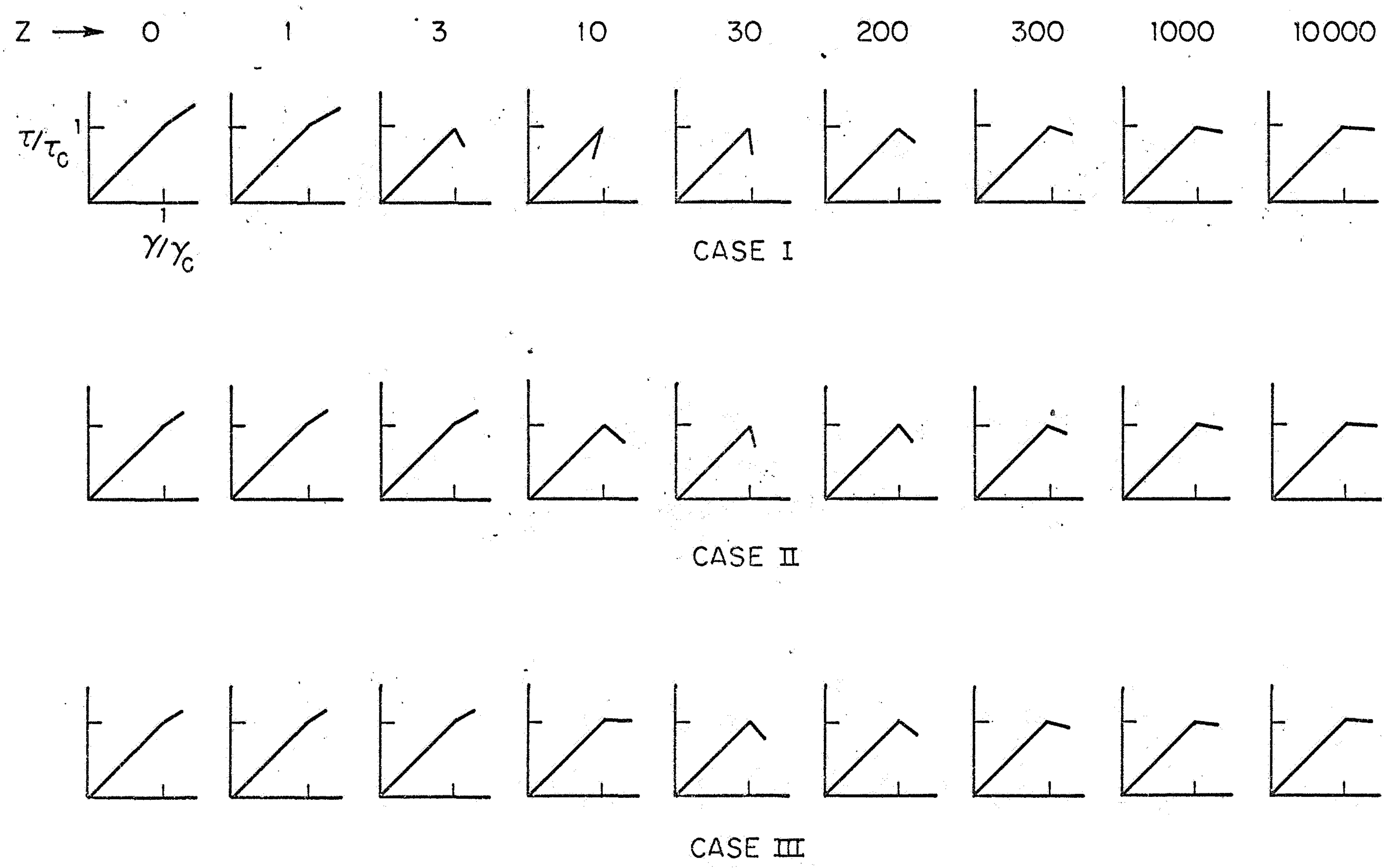

FIG. 5 INITIAL POST-BUCKLING RELATIONS BETWEEN τ/τ_c AND γ/γ_c

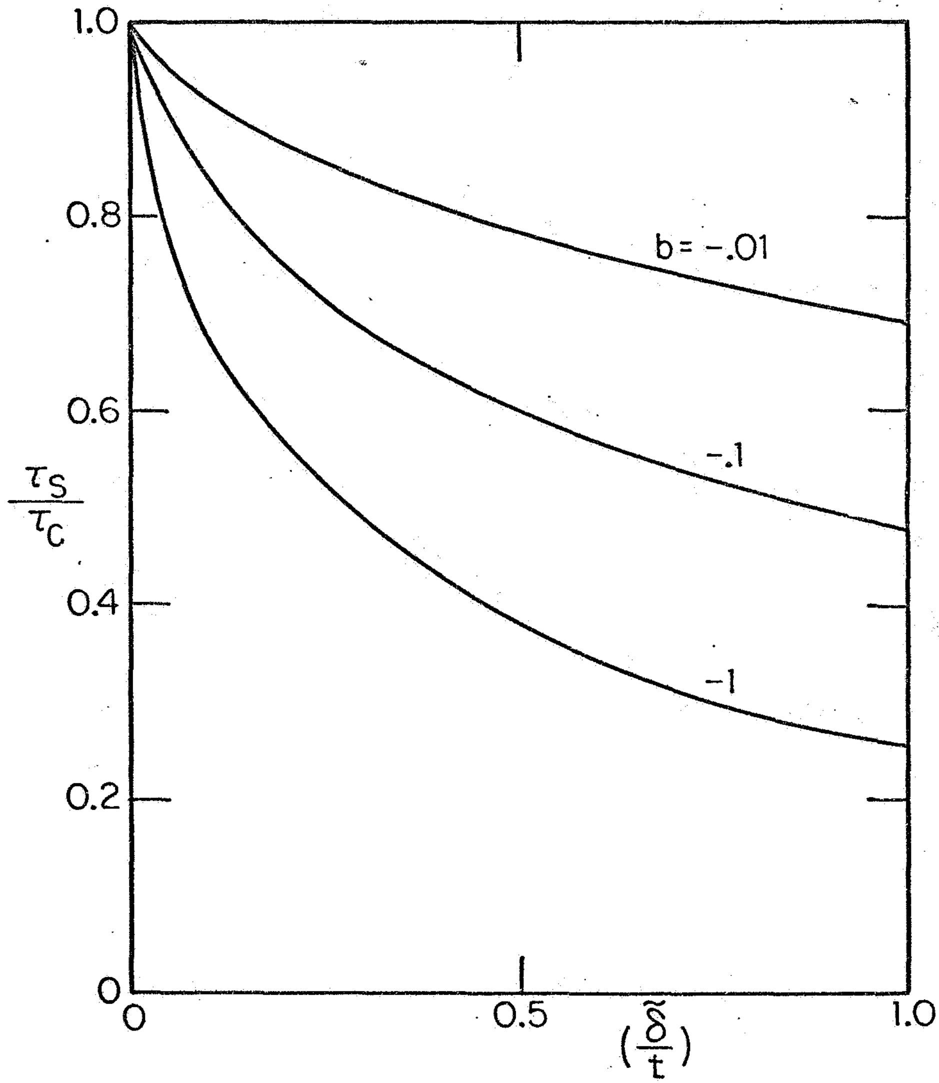

FIG. 6 DEPENDENCE OF τ_S/τ_C ON INITIAL IMPERFECTION AND ON b

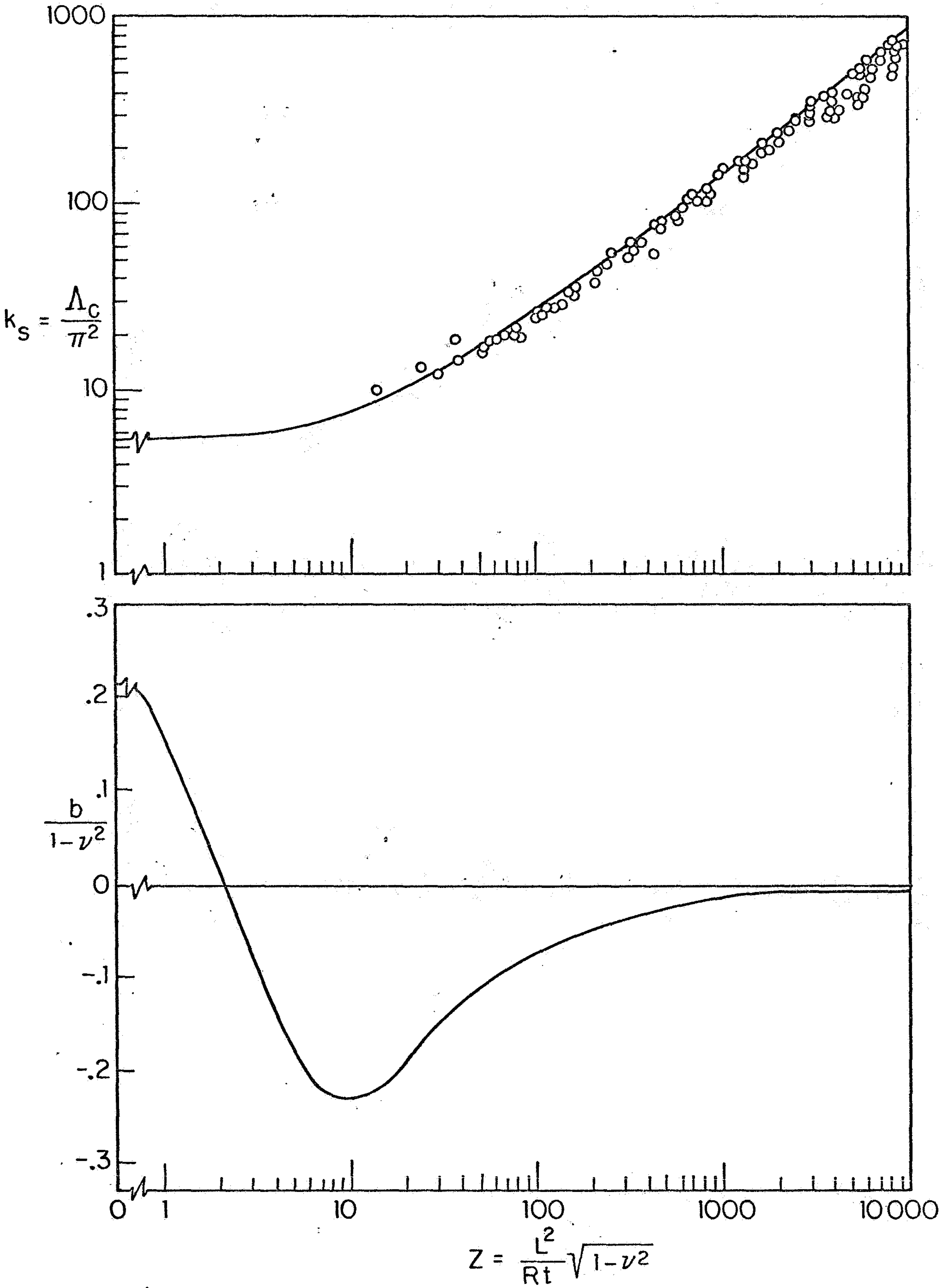

FIG. 7 COMPARISON OF THEORY AND EXPERIMENT

NSG-559 Harvard SC-7978

ERRATA

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POST-BUCKLING BEHAVIOR OF CYLINDERS IN TORSION

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Equation (30) should read

$$\Delta = \iint_A \overset{o}{N}{}^{\alpha\beta} e_{\alpha\beta} \, dA \qquad (30)$$

The phrase preceding Eq. (33) should read

"Substitution of (13) into (30), and the use of (14) and (19) leads to"